

# On a class of shift-invariant subspaces of the Drury-Arveson space

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## Abstract

In the Drury-Arveson space, we consider the subspace of functions whose Taylor coefficients are supported in the complement of a set  $Y \subset \mathbb{N}^d$  with the property that  $Y + e_j \subset Y$  for all  $j = 1, \dots, d$ . This is an easy example of shift-invariant subspace, which can be considered as a RKHS in its own right, with a kernel that can be explicitly calculated. Moreover, every such a space can be seen as an intersection of kernels of Hankel operators, whose symbols can be explicitly calculated as well. Finally, this is the right space on which Drury's inequality can be optimally adapted to a sub-family of the commuting and contractive operators originally considered by Drury.

## 1 Introduction

The Drury-Arveson space is the space  $H_d$  of functions  $f(z) = \sum_{n \in \mathbb{N}^d} a(n) z^n$  holomorphic on the unit ball in  $\mathbb{C}^d$ , such that  $\|f\|_{H_d}^2 := \sum_{n \in \mathbb{N}^d} |a(n)|^2 \beta(n)^{-1} < \infty$ , where the weight function  $\beta : \mathbb{N}^d \rightarrow \mathbb{N}$  is given by  $\beta(n) = |n|!/n!$ .

This function space was first introduced by Drury in [2], then further developed in [1]. See also [6]. It naturally arises as the right space to consider when trying to generalize to tuples of commuting operators a notable result by Von Neumann, saying that for any linear contraction  $A$  on a Hilbert space and any complex polynomial  $Q$ , it holds

$$\|Q(A)\| \leq \|Q\|_{\mathcal{M}(H^2)},$$

where  $\mathcal{M}(H^2) = H^\infty$  denotes the multiplier space of the Hardy space of the unit disc  $H^2$ . In fact, Drury shows that for a  $d$ -tuples of operators  $A = (A_1, \dots, A_d) : H \rightarrow H^d$ ,  $d \geq 2$ , such that  $[A_i, A_j] = 0$  and  $\|A\| \leq 1$ , it holds

$$\|Q(A)\| \leq \|Q\|_{\mathcal{M}(H_d)}.$$

The map  $T$  given by  $(Tg)(z) := \sum_{n \in \mathbb{N}^d} g(n) \beta(n) z^n$  defines an isometric isomorphism from  $\ell^2(\mathbb{N}^d, \beta)$  to  $H_d$ . This correspondence in particular tells us that the shift operator  $S_j g(n) = \chi_{\mathbb{N}^d + e_j}(n) g(n - e_j) \beta(n - e_j) \beta(n)^{-1}$  on  $\ell^2(\mathbb{N}^d, \beta)$  and the multiplication operator  $M_j f(z) := z_j f(z)$  on  $H_d$  are unitarily equivalent, i.e. it turns out that  $M_j T = T S_j$  for all  $j = 1, \dots, d$ .

Now, we are interested in considering subspaces of  $H_d$  of functions having Taylor coefficients with a prescribed support. Given some subset  $X$  of  $\mathbb{N}^d$ , we write  $\ell^2(X, \beta)$  for the closed subspace of  $\ell^2(\mathbb{N}^d, \beta)$  of functions supported in  $X$ . In particular, we consider subsets  $X$  with the property that

$$(1) \quad X^C + e_j \subset X^C \quad \text{for all } j = 1, \dots, d,$$

where we write  $X^C$  for the complement of  $X$  in  $\mathbb{N}^d$ . We call each such a set  $X$  a *monotone set*. Given  $g \in \ell^2(X^C, \beta)$ , for  $n \in X$  we have  $S_j g(n) = 0$  since  $n - e_j \in X$  as well. Therefore  $\ell^2(X^C, \beta)$  is a shift-invariant subspace of  $\ell^2(\mathbb{N}^d, \beta)$ . To any such a set  $X$ , we can associate the space  $H_d(X)$  of functions of  $H_d$  whose Taylor coefficients vanish on  $X^C$ . Since  $M_j T \ell^2(X^C, \beta) = T S_j \ell^2(X^C, \beta)$ , it follows that  $H_d(X^C)$  is a shift-invariant subspace of  $H_d$ .

## 2 Hankel operators

Shift-invariant subspaces for the Drury-Arveson space are characterized in [4], where it is shown that they can be represented as at most numerable intersections of kernels of Hankel operators, to be defined shortly. See also the PhD thesis [7].

Consider a Hilbert space  $\mathcal{H}$  of holomorphic functions on the unit ball  $\mathbb{B}$  of  $\mathbb{C}^d$ , such that functions holomorphic on  $\overline{\mathbb{B}}$  are dense in it. The function  $b \in \mathcal{H}$  is a symbol if there exists  $C > 0$  such that

$$|\langle fg, b \rangle_{\mathcal{H}}| \leq C \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}} \quad \text{for all } f, g \in \text{Hol}(\overline{\mathbb{B}}).$$

Endowing the space  $\overline{\mathcal{H}} := \{\bar{f} : f \in \mathcal{H}\}$  with the inner product  $\langle \bar{f}, \bar{g} \rangle_{\overline{\mathcal{H}}} := \langle g, f \rangle_{\mathcal{H}}$ , we say that  $H_b : \mathcal{H} \rightarrow \overline{\mathcal{H}}$  is a Hankel operator with symbol  $b \in \mathcal{H}$  if there exists  $C > 0$  such that

$$\langle H_b f, \bar{g} \rangle_{\overline{\mathcal{H}}} = \langle fg, b \rangle_{\mathcal{H}} \quad \text{for } f, g \in \text{Hol}(\overline{\mathbb{B}}).$$

On  $H_d$ , consider the Hankel operator with symbol  $b(z) = z^m$ , for some  $m \in \mathbb{N}^d$ . We have  $f \in \ker H_b$  iff  $\langle fg, b \rangle = 0$  for all  $g \in \text{Hol}(\overline{\mathbb{B}})$ . Since,

$$\langle fg, b \rangle_{H_d} = \widehat{fg}(m) \beta(m) = \left( \sum_{n,k} \widehat{f}(k) \widehat{g}(n) z^{n+k} \right)^{\wedge}(m) \beta(m) = \beta(m) \sum_k \widehat{f}(k) \widehat{g}(m-k),$$

it follows that  $f \in \ker H_b$  iff  $\widehat{f}(k) = 0$  for  $k \leq m$ , i.e.  $\widehat{f} \equiv 0$  on the rectangle  $R_m = \{n \in \mathbb{N}^d : n_j \leq m_j \ \forall j\}$ . Hence,  $f \in H_d(X^C)$  with  $X = R_m$ . This is the easiest example of shift-invariant subspace of the Drury-Arveson space with explicit symbol.

Actually, each set  $X$  satisfying (1) can be associated to a collection of Hankel symbols. Observe that  $X$  is bounded if and only if for all  $j$  there exists  $n \in X^C$  such that  $n \in \mathbb{N} e_j$ . In such a case,  $X$  is a finite union of rectangles,  $X = \bigcup_{k=1, \dots, K} R_{m_k}$  and hence,

$$H_d(X^C) = \bigcap_{k=1, \dots, K} \ker H_{z^{m_k}}.$$

If  $X$  is unbounded, then for every  $j$  such that  $X^C \cap \mathbb{N}e_j = \emptyset$ , we have an increasing sequence of rectangles covering the strip unbounded in the  $j$ -th direction. Summing up, it follows that

$$H_d(X^C) = \bigcap_{k=1}^{\infty} \ker H_{z^{m_k}}.$$

### 3 Drury's inequality

Let  $B_j = S_j^*$  denote the backwards shift operator on  $\ell^2(\mathbb{N}^d, \beta)$ , given by  $B_j g(n) = g(n + e_j)$ . We consider the  $d$ -tuple of operators

$$B^X = (B_1^X, \dots, B_d^X) : \ell^2(X, \beta) \rightarrow \ell^2(X, \beta)^d,$$

where for each  $j = 1, \dots, d$ ,

$$B_j^X = P_X B_j|_{\ell^2(X, \beta)},$$

being  $P_X$  the orthogonal projection of  $\ell^2(\mathbb{N}^d, \beta)$  onto  $\ell^2(X, \beta)$ . In other words,  $B_j^X$  is the compression of the standard  $j$ -th-backwards shift operator  $B_j$  to  $\ell^2(X, \beta)$ .

Observe that the adjoint of  $B^X$  is a *row contraction* from  $\ell^2(X, \beta)^d$  to  $\ell^2(X, \beta)$ ,

$$(B^X)^*(g_1, \dots, g_d) = \sum_j (B_j^X)^* g_j.$$

**Theorem 3.1.** *Let  $H$  be an abstract Hilbert space and  $A = (A_1, \dots, A_d) : H \rightarrow H^d$ ,  $d \geq 2$  a  $d$ -tuple of operators such that*

$$(i) \quad A_i A_j = A_j A_i \quad \text{for } i, j = 1, \dots, d.$$

$$(ii) \quad \|Ah\|_{H^d} \leq \|h\|_H \quad \text{for all } h \in H.$$

*Let  $X$  be the complement in  $\mathbb{N}^d$  of the set  $N := \{n \in \mathbb{N}^d : A^n = 0\}$ . Then for every complex polynomial  $Q$  of  $d$  variables, we have*

$$\|Q(A)\| \leq \|Q(B^X)\| \leq \inf\{\|R\|_{\mathcal{M}(H_d)} : R \text{ polynomial}, R(B^X) = Q(B^X)\}.$$

*Proof.* For  $N = \emptyset$  we have  $X = \mathbb{N}^d$  and this is just Drury's theorem, while for  $N = \mathbb{N}^d$ ,  $A$  reduces to a  $d$ -tuple of zeros. So, we suppose that  $N$  (and hence  $X$ ) is a proper subspace of  $\mathbb{N}^d$ .

We write  $\tilde{H}(X)$  for the space  $\ell^2(X, \tilde{H}, \beta)$ , where  $\tilde{H}$  has the same underlying space as  $H$  but a different norm,  $\|h\|_{\tilde{H}} = \|Dh\|_H$ , where  $D$  is the defect operator of  $A$ ,  $D = \sqrt{I - A^*A}$ , (see [2] for the details). Drury constructs an injective isometry  $\theta : H \rightarrow \tilde{H}(\mathbb{N}^d)$ ,  $\theta h(n) := A^n h$ , and shows that  $\tilde{B}^m \theta = \theta A^m$  for all  $m \in \mathbb{N}^d$  (here  $\tilde{B}$  is the  $d$ -tuple of backshifts on  $\tilde{H}(\mathbb{N}^d)$ ).

We rephrase this according to our setting. Let  $\pi_X$  be the orthogonal projection of  $\tilde{H}(\mathbb{N}^d)$  onto  $\tilde{H}(X)$ ,  $\tilde{B}_j^X := \pi_X \tilde{B}_j|_{\tilde{H}(X)}$  and  $\psi := \pi_X \circ \theta$ .

Since  $\theta$  is an isometry, it is easy to see that that

$$(2) \quad \psi \text{ is an isometry} \iff \theta h = 0 \text{ on } X^C \iff A^n = 0 \text{ for } n \in X^C.$$

We have

$$\psi A_j = \pi_X \tilde{B}_j \theta, \quad \text{and} \quad \tilde{B}_j^X \psi = \pi_X \tilde{B}_j|_{\tilde{H}(X)} \pi_X \theta = \pi_X \tilde{B}_j \pi_X \theta.$$

For  $n \in X$  and  $h \in H$  it holds

$$(\tilde{B}_j - \tilde{B}_j \pi_X) \theta h(n) = \theta h(n + e_j) - \pi_X \theta h(n + e_j) = \begin{cases} 0 & n + e_j \in X \\ \theta h(n + e_j) & n + e_j \notin X \end{cases}$$

which equals zero by (2). It follows that

$$\psi A^m = (\tilde{B}^X)^m \psi \quad \text{for all } m \in \mathbb{N}.$$

Hence, Drury argument can be reproduced to show that for every complex polynomial  $Q$  it holds

$$(3) \quad \|Q(A)\| \leq \|Q(B^X)\|.$$

Again following Drury's argument, it is clear that the map  $T$  given by  $(Tg)(z) := \sum_{n \in X} g(n) \beta(n) z^n$  defines an isometric isomorphism from  $\ell^2(X, \beta)$  to  $H_d(X)$ , where the latter is the space of functions  $f(z) = \sum_{n \in X} a(n) z^n$  holomorphic on the unit ball in  $\mathbb{C}^d$  such that  $\sum_{n \in X} |a(n)|^2 \beta(n)^{-1} < \infty$ . It is clear that this space has a reproducing kernel which is given by the orthogonal projection of the Drury-Arveson kernel onto  $H_d(X)$ .

Now, consider the multiplication operator  $M_j f(z) := z_j f(z)$ , and define its compression  $M_j^X = P_X M_j|_{H_d(X)}$ ,  $P_X$  being in this context the orthogonal projection from  $H_d$  onto  $H_d(X)$ . It can be shown that  $M_j^X T = T(B_j^X)^*$ , i.e. the multiplication operator  $M_j^X$  is unitary equivalent to  $(B_j^X)^*$ . It is natural to define a row contraction  $M^X : (f_1, \dots, f_d) \mapsto \sum_j M_j^X f_j$  from  $H_d^d(X)$  to  $H_d$ . However, given a complex polynomial  $Q$  of  $d$  variables, we write  $Q(M^X)$  for  $Q(M_1^X, \dots, M_d^X)$ . As in [2], it can be shown that  $\|Q(B^X)\| = \|Q(M^X)\|$ . Therefore, inequality (3) can be re-written as

$$\|Q(A)\| \leq \|Q(M^X)\|.$$

For  $f \in H_d(X)$ , we have  $M_j^X f(z) = \sum_{n \in X \cap X + e_j} a_{n - e_j} z^n$ , and so

$$\|M_j^X f\|_{H_d}^2 = \sum_{n \in X \cap X - e_j} |a_n|^2 \beta(n + e_j)^{-1} \leq \sum_{n \in X \cap X - e_j} |a_n|^2 \beta(n)^{-1} \leq \|f\|_{H_d}^2.$$

It follows that all polynomials are multipliers for  $H_d$  and

$$\|Q(M^X)\| = \|P_X Q(M)\| \leq \|Q(M)\| = \|Q\|_{\mathcal{M}(H_d)}.$$

Of course, there are many polynomials  $R$  such that  $P_X R(M) = P_X Q(M)$ , so we can finally write

$$(4) \quad \|Q(A)\| \leq \|Q(B^X)\| \leq \inf\{\|R\|_{\mathcal{M}(H_d)} : R \text{ polynomial}, R(B^X) = Q(B^X)\}.$$

□

Observe that the first inequality in the theorem is optimal if the backshift  $d$ -tuple  $B^X$  is satisfies (i) and (ii) and if  $\{n \in \mathbb{N}^d : (B^X)^n = 0\}$  equals  $N$ . It is clear that condition (ii) holds for

$B^X$ , for every choice of  $X$ . Also,  $n \in N \iff n + m \in N$  for all  $m \in \mathbb{N}^d$  and since  $N = X^C$  this is equivalent as asking  $f(n + m) = 0$  for all  $m \in \mathbb{N}^d$ ,  $f \in \ell^2(X, \beta)$ . But  $f(n + m) = (B^X)^n f(m)$  and so  $\{n \in \mathbb{N}^d : (B^X)^n = 0\} = N$ .

On the other hand, the commuting property (i) is not fulfilled on most sets  $X$ . Of course, if  $X$  is chosen such that  $\ell^2(X, \beta)$  is backshift-invariant, then  $B^X = B|_{\ell^2(X, \beta)}$  and (i) and (ii) hold, see [2]. More in general, doing standard calculations it is not hard to see that  $B^X$  satisfies (i) if and only if

$$(5) \quad n, n + e_i + e_j, n + e_i \in X \implies n + e_j \in X, \quad \text{for } i, j = 1, \dots, d.$$

This is a shape-condition on the set  $X$ , saying that it cannot have any subset with one of the following configurations



Figure 1: Fat dots are elements of not permitted subsets of  $X$ .

It is clear that  $X = N^C$  satisfies (5), since  $x + e_j \in X^C$  for some  $j$  would imply  $x + e_j + e_i \in X^C$  for all  $i = 1, \dots, d$ . It follows that the inequality in the theorem is optimal.

**An open problem.** Does equality hold in the second inequality of the theorem? The reason to be optimistic in this sense comes from a theorem proved by Sarason in [5] (see also [3, Theorem 3.1]) in the one-dimensional case, i.e. for the Hardy space, which can possibly be extended to be true on the Drury-Arveson space. Let  $K$  be a closed backshift-invariant subspace of the Hardy space  $H^2$ , and write  $S_K$  for the compression of the shift operator to this subspace. Sarason proved the following.

**Theorem 3.2.** *Let  $T$  be an operator commuting with  $S_K$ . Then there exists a function  $\varphi \in H^\infty$  such that  $T = \varphi(S_K)$  and  $\|T\| = \|\varphi\|_{H^\infty}$ .*

Now, on  $H_d$  the operator  $T = Q(M^X)$  clearly commutes with  $M_j^X$ , for all  $j$ , so one would like to conclude that there exists a function  $\varphi_j$  in the multiplier space of  $H_d$ , possibly different from the polynomial  $Q$ , such that  $T = \varphi_j(M_j^X)$  and  $\|T\| = \|\varphi_j\|_{\mathcal{M}_{H_d}}$ . Since this holds for every  $j = 1, \dots, d$ , it is tempting to think that there exists a polynomial  $R$  of  $d$  variables such that  $T = R(M^X)$  and  $\|T\| = \|R\|_{\mathcal{M}_{H_d}}$ . If this is the case, (4) would become

$$\|Q(A)\| \leq \|Q(B^X)\| = \|Q(M^X)\| = \|R\|_{\mathcal{M}_{H_d}},$$

where  $R$  is some polynomial such that  $R(M^X) = Q(M^X)$ .

## 4 A closed formula for the reproducing kernel on slabs

The Drury-Arveson space has a reproducing kernel. For  $f \in H_d$  and  $z \in \mathbb{D}$ , we have

$$f(z) = \sum_n a_n z^n = \sum_n a_n \frac{z^n}{\beta(n)} \beta(n) = \langle f, k_z \rangle_{H_d},$$

with  $k_z(w) = \sum_n \beta(n) \bar{z}^n w^n$  for  $w \in \mathbb{D}$ .

The series can be explicitly calculated and we get

$$k_z(w) = \sum_{n \in \mathbb{N}} \beta(n) \bar{z}^n w^n = \sum_{k \geq 0} \sum_{|n|=k} \binom{k}{n} \bar{z}^n w^n = \sum_{k \geq 0} \left( \sum_{j=1}^d \bar{z}_j w_j \right)^k = \sum_{k \geq 0} (\bar{z} \cdot w)^k = \frac{1}{1 - \bar{z} \cdot w}.$$

Now, let  $X$  be some subset of  $\mathbb{N}^d$  satisfying (1). As we have already pointed out, the reproducing kernel  $k^X(w, z)$  of  $H_d(X)$  is the orthogonal projection of the Drury-Arveson reproducing kernel onto the subspace  $H_d(X)$ , in the sense that

$$(6) \quad k^X(w, z) = P_X k(w, z) = P_X k_z(w) = \sum_{n \in X} \beta(n) \bar{z}^n w^n.$$

For some special choices of the set  $X$  we are able to get a closed formula for the reproducing kernel in (6). In particular, this can be done when  $X$  is what we call a *slab*,  $\mathcal{S}_1 = \{n \in \mathbb{N}^d : n_1 = 0, \dots, N_1\}$ .

**Proposition 4.1.** *For  $X = \mathcal{S}_1$  it holds*

$$(7) \quad k^{\mathcal{S}_1}(w, z) = \frac{1}{1 - \bar{z} \cdot w} \left( 1 - \frac{\bar{z}_1 w_1}{1 - \bar{z} \cdot w + \bar{z}_1 w_1} \right)^{N_1}.$$

*Proof.* Set  $t = \bar{z}w$ . As a first step, suppose that  $d = 2$ . Using the fact that for  $j, k \in \mathbb{N}$  it holds

$$\sum_{j=0}^{\infty} \binom{j+k}{j} x^j = \frac{1}{(1-x)^{k+1}},$$

we get

$$\begin{aligned} k^X(w, z) &= \sum_{n \in X} \beta(n) \bar{z}^n w^n = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{\infty} \binom{n_1+n_2}{n_2} t_1^{n_1} t_2^{n_2} = \sum_{n_1=0}^{N_1} t_1^{n_1} \frac{1}{(1-t_2)^{n_1+1}} \\ &= \frac{1}{1-t_2} \sum_{n_1=0}^{N_1} \left( \frac{t_1}{1-t_2} \right)^{n_1} = \frac{1}{1-t_2} \frac{1 - \left( \frac{t_1}{1-t_2} \right)^{N_1}}{1 - \frac{t_1}{1-t_2}} \\ &= \frac{1 - \left( \frac{t_1}{1-t_2} \right)^{N_1}}{1 - t_1 - t_2} = \frac{1}{1 - \bar{z} \cdot w} \left( 1 - \frac{\bar{z}_1 w_1}{1 - \bar{z} \cdot w + \bar{z}_1 w_1} \right)^{N_1}. \end{aligned}$$

Now, suppose that (7) holds on  $\mathbb{N}^{d-1}$ . Again, suppose to re-order the basis  $e_1, \dots, e_d$  so that

$j = 1$ . On  $\mathbb{N}^d$  we have

$$\begin{aligned}
k^X(w, z) &= \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \binom{(|n| - n_d) + n_d}{n_d} \frac{(n_1 + \cdots + n_{d-1})!}{n_1! \cdots n_{d-1}!} t_1^{n_1} \cdots t_d^{n_d} \\
&= \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{\infty} \cdots \sum_{n_{d-1}=0}^{\infty} \frac{(n_1 + \cdots + n_{d-1})!}{n_1! \cdots n_{d-1}!} t_1^{n_1} \cdots t_{d-1}^{n_{d-1}} \frac{1}{(1 - t_d)^{n_1 + \cdots + n_{d-1} + 1}} \\
&= \frac{1}{1 - t_d} \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{\infty} \cdots \sum_{n_{d-1}=0}^{\infty} \frac{(n_1 + \cdots + n_{d-1})!}{n_1! \cdots n_{d-1}!} \left( \frac{t_1}{(1 - t_d)} \right)^{n_1} + \cdots + \left( \frac{t_{d-1}}{(1 - t_d)} \right)^{n_{d-1}} \\
&= \frac{1}{1 - t_d} \frac{1}{1 - \sum_{i=1}^{d-1} \frac{t_i}{(1 - t_d)}} \left( 1 - \frac{\frac{t_1}{1 - t_d}}{1 - \sum_{i=2}^{d-1} \frac{t_i}{(1 - t_d)}} \right)^{N_1} \\
&= \frac{1}{1 - \sum_{i=1}^d t_i} \left( 1 - \frac{1}{1 - \sum_{i=2}^d t_i} \right)^{N_1} = \frac{1}{1 - \bar{z} \cdot w} \left( 1 - \frac{\bar{z}_j w_j}{1 - \bar{z} \cdot w + \bar{z}_1 w_1} \right)^{N_1}.
\end{aligned}$$

□

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